

Orthogonal polynomials on the disk in the absence of finite moments

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Abstract

We introduce a new family of orthogonal polynomials on the disk that has emerged in the context of wave propagation in layered media. Unlike known examples, the polynomials are orthogonal with respect to a measure all of whose even moments are infinite.

1 Introduction

For each $\alpha > -1$ there is a corresponding family of disk polynomials that are orthogonal with respect to the measure $(1 - x^2 - y^2)^\alpha dx dy$ on the unit disk \mathbb{D} ; these are sometimes referred to as generalized Zernike polynomials, named for the case $\alpha = 0$ introduced in [8]. The well-established theory of disk polynomials is detailed in [4, 6, 1]. The constraint $\alpha > -1$ stems from the requirement that the measure $(1 - x^2 - y^2)^\alpha dx dy$ have finite moments, which is necessary for meaningful evaluation of the corresponding scalar product

$$\langle p, q \rangle_\alpha = \int_{\mathbb{D}} p(x, y) \overline{q(x, y)} (1 - x^2 - y^2)^\alpha dx dy \quad (1.1)$$

on arbitrary polynomials p and q . Recent work on the propagation of waves in layered media [2, 3] has brought to light a family of polynomials orthogonal with respect to $(1 - x^2 - y^2)^{-1} dx dy$. Since

$$\int_{\mathbb{D}} x^{2m} y^{2n} (1 - x^2 - y^2)^{-1} dx dy = \infty \quad (1.2)$$

for every pair of nonnegative integers m and n , the scalar product (1.1) is not defined for arbitrary polynomials in the case $\alpha = -1$. Nevertheless, polynomials—which we term scattering polynomials—comprise an orthogonal basis for $L^2(\mathbb{D}, dx dy / (1 - x^2 - y^2))$. The purpose of the present paper is to present the details of this result.

2 Definition and properties of scattering polynomials

Referring to the notation $z = x + iy$ for points in the unit disk \mathbb{D} , one has the option of working with euclidean x, y -coordinates, or with complex coordinates z and \bar{z} . As far as orthogonal polynomials are concerned these are essentially equivalent, as elaborated in [7]; the present paper uses whichever coordinates are most convenient for the task at hand.

We define *scattering polynomials* by a Rodrigues type formula, as follows. For every $(p, q) \in \mathbb{Z}^2$ with $\min\{p, q\} \geq 1$, set

$$\varphi^{(p, q)}(z) = \frac{(-1)^p}{q(p + q - 1)!} (1 - z\bar{z}) \frac{\partial^{p+q}}{\partial z^p \partial \bar{z}^q} (1 - z\bar{z})^{p+q-1}. \quad (2.1)$$

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The chosen normalization simplifies the formulation of the boundary Green's function for scattering in layered media (see [3]) and so is physically natural, although not important for present considerations.

Note that disk polynomials satisfy a Rodrigues formula similar to that of scattering polynomials, but there is a qualitative difference: it follows directly from (2.1) that $\varphi^{(p,q)}(z) = 0$ for every z on the unit circle \mathbb{T} , whereas all disk polynomials have constant non-zero modulus on \mathbb{T} , cf. [4]. Our main result concerns completeness of scattering polynomials, as follows.

Theorem 1 *Scattering polynomials $\varphi^{(p,q)}$ defined by (2.1), where $(p, q) \in \mathbb{Z}^2$ and $\min\{p, q\} \geq 1$, comprise an orthogonal basis for $L^2(\mathbb{D}, dxdy/(1 - x^2 - y^2))$.*

In §2.1 and §2.2 below we show that scattering polynomials are eigenfunctions of a second order differential operator and may be expressed in terms of Jacobi polynomials; these results contribute to a proof of Theorem 1 completed in §2.3.

2.1 Eigenfunctions of $-(1 - x^2 - y^2)\Delta/4$

Let $\tilde{\Delta}$ denote the modified laplacian

$$\tilde{\Delta} = (1 - z\bar{z})\frac{\partial^2}{\partial z\partial\bar{z}} = \frac{1 - x^2 - y^2}{4}\Delta, \quad (2.2)$$

where Δ is the usual (euclidean) laplacian. Direct computation using (2.1) shows that for all integers $p, q \geq 1$,

$$-\tilde{\Delta}\varphi^{(p,q)} = pq\varphi^{(p,q)}. \quad (2.3)$$

Letting $\sigma_0 : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ denote the divisor function, there is thus a family of $\sigma_0(k)$ eigenfunctions of $-\tilde{\Delta}$ of the form $\varphi^{(p,q)}$ corresponding to each positive integer eigenvalue k . We show in the next section that these eigenfunctions are linearly independent.

2.2 Representation in terms of Jacobi polynomials

Like disk polynomials, scattering polynomials have a representation in terms of Jacobi polynomials, but again, there is a qualitative difference. The disk polynomials corresponding to parameter $\alpha > -1$ can be expressed in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}$ for nonnegative integer values of β . Since there is no Jacobi polynomial corresponding to $\alpha = -1$, the same cannot be true for scattering polynomials. Indeed it turns out that scattering polynomials can be formulated in terms of $P_n^{(1,\beta)}$, where β is a nonnegative integer, as follows.

Expanding the binomial $(1 - z\bar{z})^{p+q-1}$ in the formula (2.1), and then applying the derivative $\partial^{p+q}/\partial z^p\partial\bar{z}^q$, yields

$$\varphi^{(p,q)}(z) = \frac{(-1)^{q+\nu+1}}{q}(1 - z\bar{z})z^{m+\nu-p+1}\bar{z}^{m+\nu-q+1}\sum_{j=0}^{\nu}(-1)^j\frac{(j+\nu+m+1)!}{j!(j+m)!(\nu-j)!}(z\bar{z})^j, \quad (2.4)$$

where $m = |p - q|$ and $\nu = \min\{p, q\} - 1$; the latter notation will be used in the remainder of this section. Switching to polar form $z = re^{i\theta}$, it follows from (2.4) that

$$\varphi^{(p,q)}(re^{i\theta}) = e^{i(q-p)\theta}f^{(p,q)}(r), \quad (2.5)$$

where

$$f^{(p,q)}(r) = \frac{(-1)^{q+\nu+1}}{q} (1-r^2) r^m \sum_{j=0}^{\nu} (-1)^j \frac{(j+\nu+m+1)!}{j!(j+m)!(\nu-j)!} r^{2j}. \quad (2.6)$$

The radial functions $f^{(p,q)}$ were first discovered in [2], as was the following connection to Jacobi polynomials, valid for $\nu \geq 0$:

$$f^{(p,q)}(r) = \frac{(-1)^{q+m+\nu+1}(m+\nu+1)}{q} (1-r^2) r^m P_{\nu}^{(1,m)}(2r^2-1). \quad (2.7)$$

Combined with (2.5) this yields the representation

$$\varphi^{(p,q)}(re^{i\theta}) = \frac{(-1)^{q+\max\{p,q\}} \max\{p,q\}}{q} (1-r^2) r^{|p-q|} P_{\min\{p,q\}-1}^{(1,|p-q|)}(2r^2-1) e^{i(q-p)\theta}. \quad (2.8)$$

Note that the angular part of $\varphi^{(p,q)}(re^{i\theta})$, namely $e^{i(q-p)\theta}$, is a pure frequency. Therefore if $q-p \neq q'-p'$, then $\varphi^{(p,q)}$ and $\varphi^{(p',q')}$ are orthogonal in

$$L^2\left(\mathbb{D}, \frac{dxdy}{1-x^2-y^2}\right) = L^2\left(\mathbb{D}, \frac{rdrd\theta}{1-r^2}\right).$$

In particular, if $pq = p'q'$ and $(p,q) \neq (p',q')$, then $\varphi^{(p,q)}$ and $\varphi^{(p',q')}$ are orthogonal, so the set of scattering polynomials corresponding to any fixed eigenvalue of $-\tilde{\Delta}$ is linearly independent.

2.3 Completeness in $L^2\left(\mathbb{D}, \frac{rdrd\theta}{1-r^2}\right)$

In general, given a measure μ on a locally compact metric space X and a positive measurable weight function $w : X \rightarrow \mathbb{R}_+$,

$$L^2(X, w d\mu) = \frac{1}{\sqrt{w}} L^2(X, d\mu), \quad (2.9)$$

and a sequence $\{b_{\nu}\}_{\nu=0}^{\infty}$ is an orthogonal basis for $L^2(X, w d\mu)$ if and only if the corresponding sequence $\{\sqrt{w}b_{\nu}\}_{\nu=0}^{\infty}$ is an orthogonal basis for $L^2(X, d\mu)$. In particular, setting $d\mu = rdrd\theta/(1-r^2)$,

$$L^2(\mathbb{D}, d\mu) = \sqrt{1-r^2} L^2(\mathbb{D}, rdrd\theta). \quad (2.10)$$

Also, since for any nonnegative integer m , $\{P_{\nu}^{(1,m)}(u)\}_{\nu=0}^{\infty}$ is an orthogonal basis for

$$L^2([-1, 1], (1-u)(1+u)^m du),$$

it follows that the quasipolynomials

$$Q_{\nu}^{(1,m)}(u) = \left(\frac{1-u}{2}\right)^{\frac{1}{2}} \left(\frac{1+u}{2}\right)^{\frac{m}{2}} P_{\nu}^{(1,m)}(u) \quad (2.11)$$

comprise an orthogonal basis for $L^2([-1, 1], du)$; see [5].

In order to show that

$$\mathcal{B} = \left\{ \varphi^{(p,q)} \mid (p,q) \in \mathbb{Z}^2 \text{ \& } \min\{p,q\} \geq 1 \right\} \quad (2.12)$$

is an orthogonal basis of $L^2(\mathbb{D}, r dr d\theta / (1 - r^2))$, we first argue that the functions $\varphi^{(p,q)}$ are orthogonal, and then that the span of \mathcal{B} is dense. It was proven in §2.2 that $\varphi^{(p,q)}$ and $\varphi^{(p',q')}$ are orthogonal if $pq = p'q'$ and $(p, q) \neq (p', q')$. On the other hand, if $pq \neq p'q'$, then orthogonality of $\varphi^{(p,q)}$ and $\varphi^{(p',q')}$ follows from the fact that they are eigenfunctions, corresponding to distinct eigenvalues, of the self-adjoint operator $-\tilde{\Delta}$; self-adjointness of $-\tilde{\Delta}$ follows from that of $-\Delta$ by (2.2).

It remains to show that $\text{span } \mathcal{B}$ is dense in $L^2(\mathbb{D}, r dr d\theta / (1 - r^2))$. Toward this end, suppose that $h \in L^2(\mathbb{D}, r dr d\theta / (1 - r^2))$ is orthogonal to every member of \mathcal{B} . By (2.10) there exists $g \in L^2(\mathbb{D}, r dr d\theta)$ such that

$$h(r, \theta) = \sqrt{1 - r^2} g(r, \theta). \quad (2.13)$$

Let $\alpha_{p,q} = (-1)^{q+\max\{p,q\}} \max\{p,q\}/q$ denote the coefficient occurring on the right-hand side of (2.8). Then for each fixed $n = q - p \in \mathbb{Z}$, for every $\nu = \min\{p, q\} - 1 \geq 0$,

$$\begin{aligned} 0 &= \int_{\mathbb{D}} h(r, \theta) \overline{\varphi^{(p,q)}(r, \theta)} \frac{r dr d\theta}{1 - r^2} \\ &= \alpha_{p,q} \int_0^1 \left(\int_0^{2\pi} h(r, \theta) e^{-in\theta} d\theta \right) (1 - r^2) r^{|n|} P_{\nu}^{(1,|n|)}(2r^2 - 1) \frac{r dr}{1 - r^2} \quad (\text{by (2.7)}) \\ &= \alpha_{p,q} \int_0^1 \left(\int_0^{2\pi} g(r, \theta) e^{-in\theta} d\theta \right) \sqrt{1 - r^2} r^{|n|} P_{\nu}^{(1,|n|)}(2r^2 - 1) r dr \quad (\text{by (2.13)}) \\ &= \frac{\alpha_{p,q}}{4} \int_{-1}^1 \left(\int_0^{2\pi} g\left(\sqrt{\frac{1+u}{2}}, \theta\right) e^{-in\theta} d\theta \right) \sqrt{\frac{1-u}{2}} \sqrt{\frac{1+u}{2}}^{|n|} P_{\nu}^{(1,|n|)}(u) du \quad (u = 2r^2 - 1) \\ &= \frac{\alpha_{p,q}}{4} \int_{-1}^1 \left(\int_0^{2\pi} g\left(\sqrt{\frac{1+u}{2}}, \theta\right) e^{-in\theta} d\theta \right) Q_{\nu}^{(1,|n|)}(u) du \quad (\text{as in (2.11)}). \end{aligned}$$

Since the quasipolynomials $Q_{\nu}^{(1,|n|)}$ are an orthogonal basis for $L^2([-1, 1], du)$, it follows that for each $n \in \mathbb{Z}$,

$$\int_0^{2\pi} g\left(\sqrt{\frac{1+u}{2}}, \theta\right) e^{-in\theta} d\theta = 0 \quad (2.14)$$

for every $u \in [-1, 1]$ outside a set E_n of measure zero. Since $\{e^{in\theta}\}_{n \in \mathbb{Z}}$ is an orthogonal basis of $L^2([0, 2\pi], d\theta)$, it follows in turn that for $u \notin \cup E_n$

$$g\left(\sqrt{\frac{1+u}{2}}, \theta\right) = 0,$$

for almost every $\theta \in [0, 2\pi]$. Thus $g(r, \theta) = 0$ for almost every $(r, \theta) \in \mathbb{D}$ and $g = 0$ as a function in $L^2(\mathbb{D}, r dr d\theta)$, whence $h = 0$ also. This proves that the orthogonal complement of \mathcal{B} in $L^2(\mathbb{D}, r dr d\theta / (1 - r^2))$ is empty, and hence that \mathcal{B} is an orthogonal basis.

3 Conclusions

Since the vector space $L^2(\mathbb{D}, r dr d\theta / (1 - r^2)) = \sqrt{1 - r^2} L^2(\mathbb{D}, r dr d\theta)$ is dense in

$$L^2(\mathbb{D}, r dr d\theta) = L^2(\mathbb{D}, dxdy),$$

and convergence in the former space implies convergence in the latter, scattering polynomials comprise a (non-orthogonal) basis for $L^2(\mathbb{D}, dxdy)$ consistent with Dirichlet boundary values. From the

perspective of analysis of functions on the disk, this provides an alternative to Zernike polynomials—and their generalizations the disk polynomials—which are non-zero on the boundary circle and so inconsistent with Dirichlet conditions.

More generally, scattering polynomials illustrate that orthogonal polynomials can comprise an orthogonal basis for a function spaces $L^2(X, d\mu)$ in which not all polynomials are integrable. A natural question for further investigation is the existence and extent of other such examples.

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